TRIGONOMETRY

Inverse Trig Functions

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In *Graphing the trigonometric functions*, we said that a function is expressed in the form f(x) where x is the input of the function. Functions also have an inverse function where, putting the output of a regular function as the input to its inverse function returns the original input to the original function. In other words, if the sine of 30° is ½, then the inverse sine of ½ is 30°.

Inverse functions are expressed as having an exponent of -1 (inverse of f(x) is $f^{-1}(x)$, not $f(x)^{-1}$). This however does not mean sin^{-1} can be written as 1/sin, as this would be written as $(sin)^{-1}$. Seeing a unit negative exponent next to a function exclusively denotes that it is its inverse function. Usually, you can write inverse trig functions as sin^{-1}, cos^{-1}, tan^{-1}, cot^{-1}, sec^{-1}, and csc^{-1} but they can also be written with the prefix "arc" such as, arcsin, arccos, arctan, etc. One way of remembering is to denote arc as "the angle whose sine is-". For example, arcsin½ means the angle whose sine is ½, id est 30°.

$$\sin 90° = 1 \text{ so, } \arcsin 1 = 90°$$

$$\cos 60° = ½ \text{ so, } \arccos ½ = 60°$$

$$\tan 45° = 1 \text{ so, } \arctan 1 = 45°$$

Remember,

$$\sin^{-1} \neq \frac{1}{sin} \text{ and,}$$

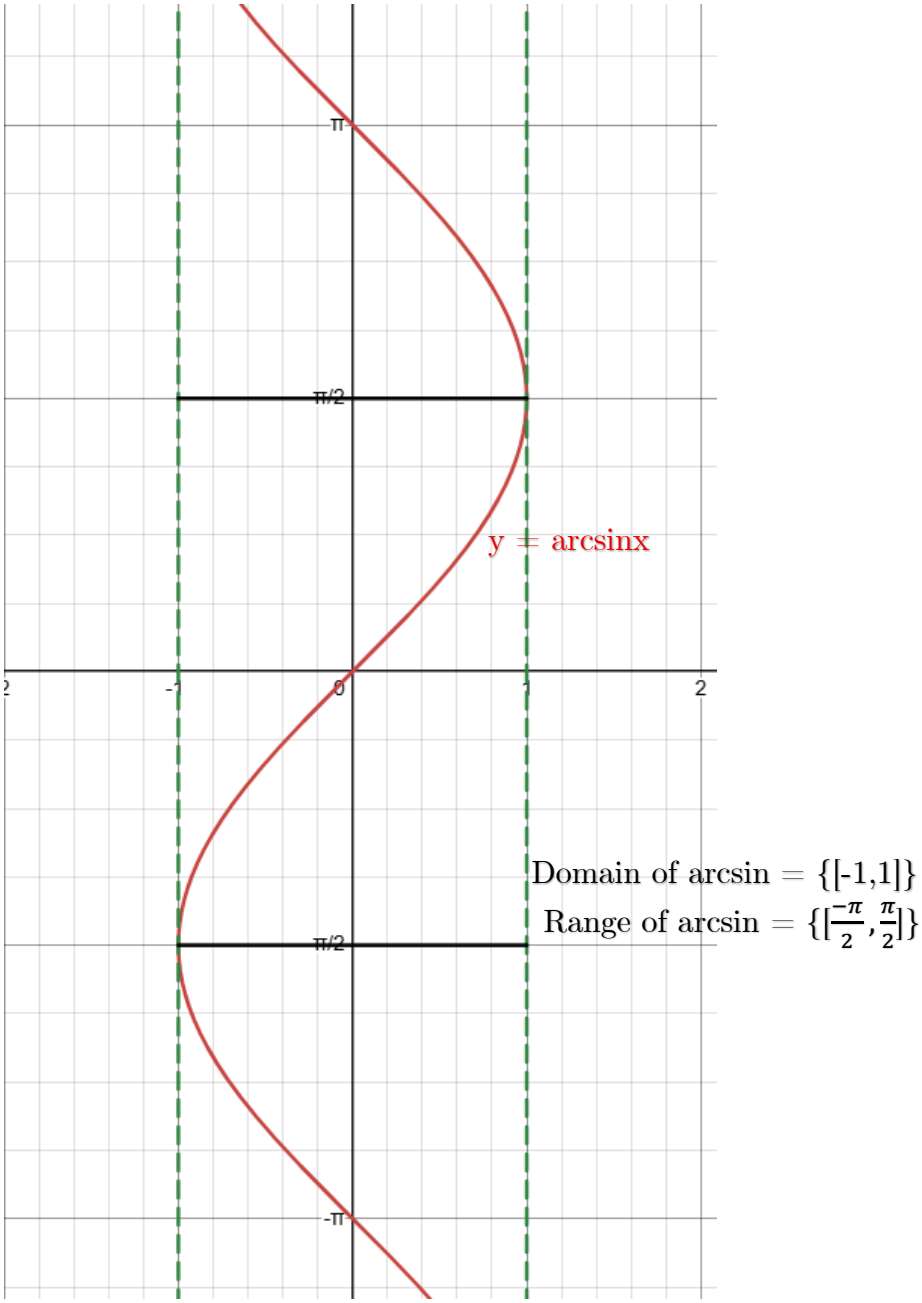
$$\sin^{-1} = \arcsin$$

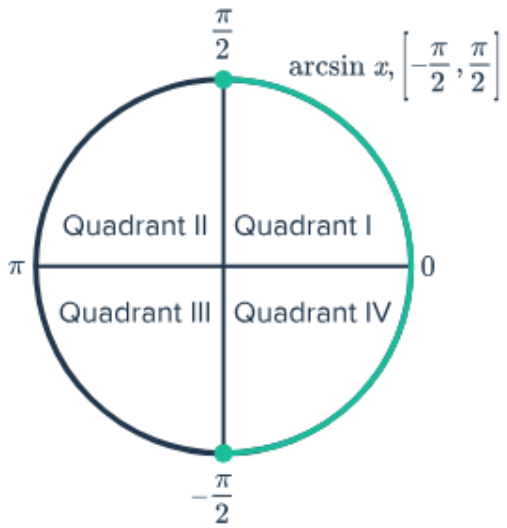
But here, one might realise that although the sine and cosine of any real number is defined, arcsin and arccos should be theoretically indefinite because there are multiple angles whose sine or cosine is the same. For example, sin30° = 1/2 , but also sin150° = ½, sin390° = ½, and so on. So why is arcsin½ = 30° and not 150° or 390°?

The reason is that, functions must always return one value for each argument. And so, by convention, the range of arcsin is set to {[-π/2, π/2]} or {[-90°, 90°]}. For arccos, its range is set to {[0, π]} or {[0, 180°]}. Before we look at the other functions, let us understand why the range is limited to these numbers specifically.

If we see the graphs of arcsin and arccos, they are simply the graphs of sine and cosine but drawn vertically that is, its reflection across the diagonal y = x. However, as we said, a function can only return one value per argument. On a graph this means a vertical column representing the output of $f^{-1}(x)$, should be able to pass one point on the function at a time yet, it does not, so we limit its range by convention.

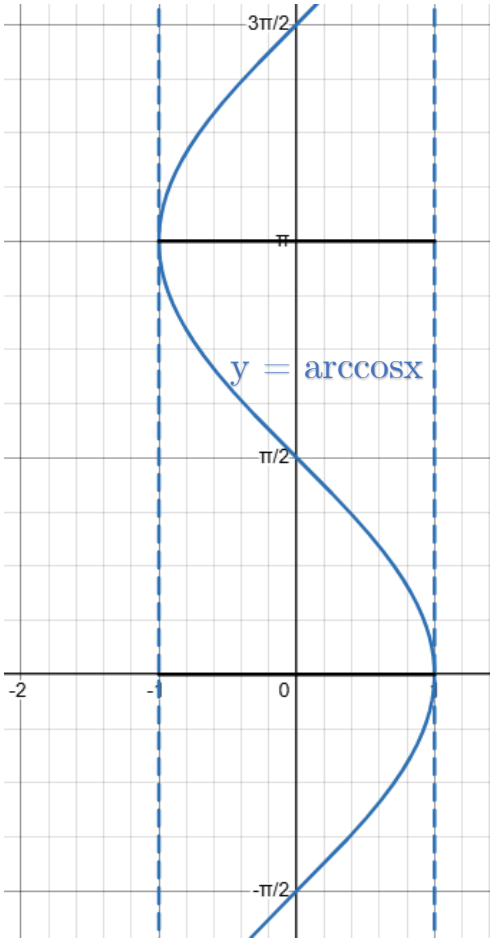
On a unit circle, the range of arcsin and arccos is drawn as two hemispheres of a circle. This is also a useful memory aid.





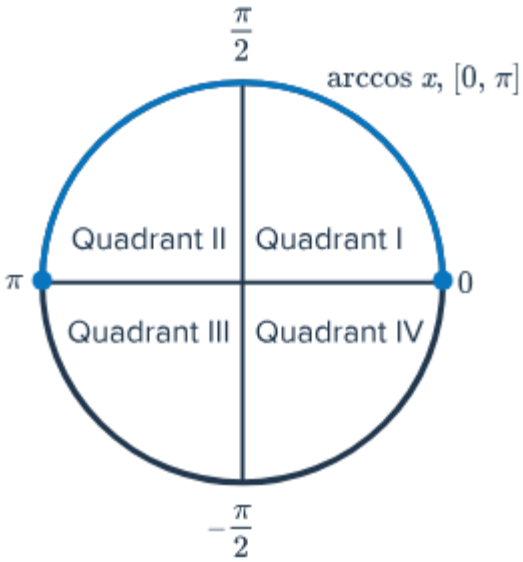
As sine can only output values from -1 to 1, no angle will have a sine outside -1 and 1 so, the domain of arcsin is {[-1,1]}.

Similarly to sine, here is the graph of arccos. While the range of arcsin is in Quadrant I and Quadrant IV, the range of arccos is in Quadrant I to Quadrant II or in other words, {[0, π]}. This is also means only arcsin can return negative angles (in Quadrant IV).

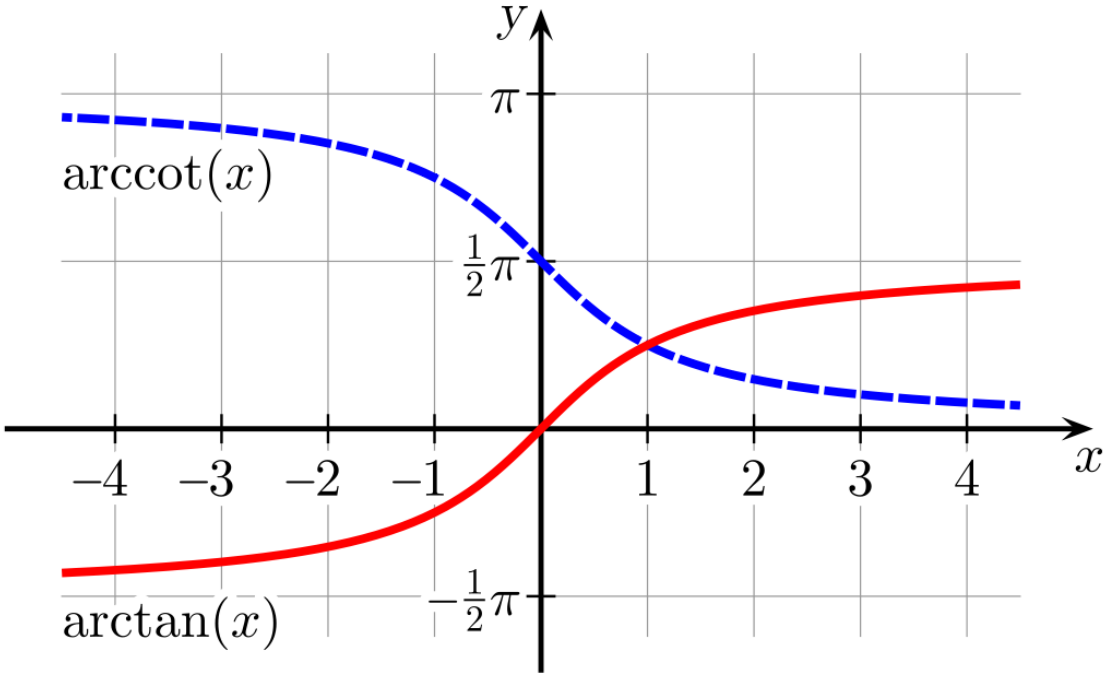


Domain of arccos = {[-1, 1]}

Range of arccos = {[0, π]}



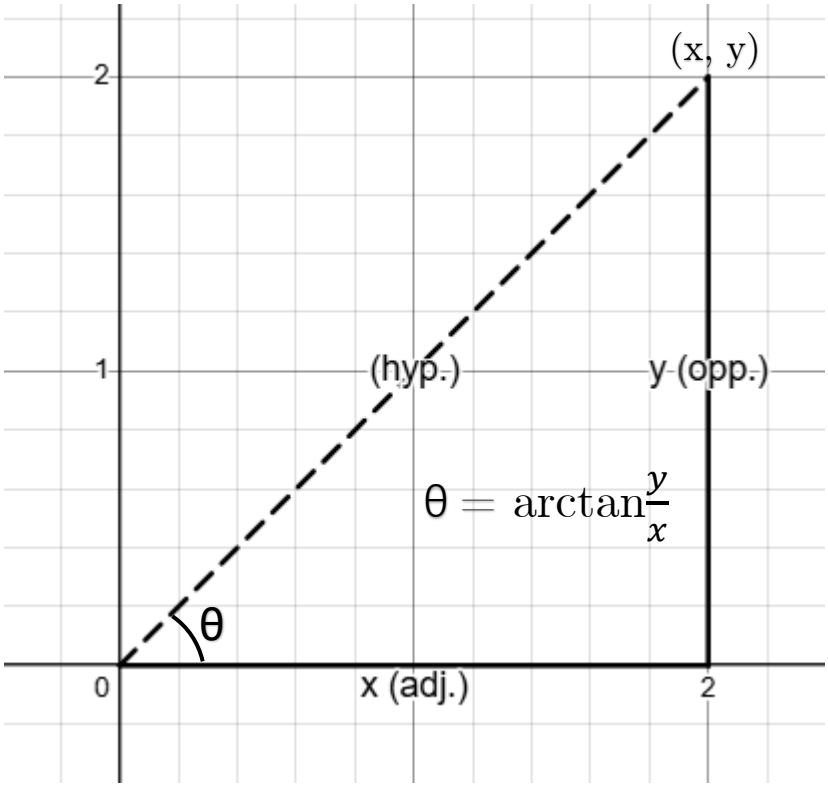
Now moving on, we have the arctan function which is the inverse function of tangent. Looking at its graph, it is simply the tanθ function flipped by 90° or π/2, like arcsin and arccos, with its range restricted to {(-π/2, π/2)}, same as arcsin but excluding π/2 and -π/2. Also as expected, the domain of arctan is the range of tangent which is {(-∞, ∞)}. Arccot is also included here and, its range lies from {(0, π)}, same as arccos but excluding 0 and π.



arctan	arccot
Domain - {(-∞, ∞)}	Domain - {(-∞, ∞)}
Range - {(-π/2, π/2)}	Range - {(0, π)}

**exclusive of -π/2, π/2, 0, and π respectively

The function arctan is commonly used in coordinate systems to find the angle of a line from the origin to a point (x, y). That is, the angle of (x, y) relative to the origin. We can do this because the x length, the y length and the hypotenuse (the line from the origin to x, y) form a right-angled triangle. And, since $\tan\theta = \frac{opp.}{adj.} = \frac{y}{x}$ we can isolate θ using $\arctan\frac{y}{x}$.



Moving forward, we have the arcsec and arccsc functions, id est, the inverse functions of secant and cosecant. However, they are not used as much as the other functions but should still be known when learning trig. The important thing is to understand and also derive the graphs included here, from there you will be able to memorise the range and domain of these functions, and so you will also be able to account for them. Since, for example, arcsinx = θ will never return angles in Quadrant II or III, you can subtract θ from 180° to find its supplementary angle in Quadrant II or Quadrant III. This works because two supplementary angles have the same sine value.

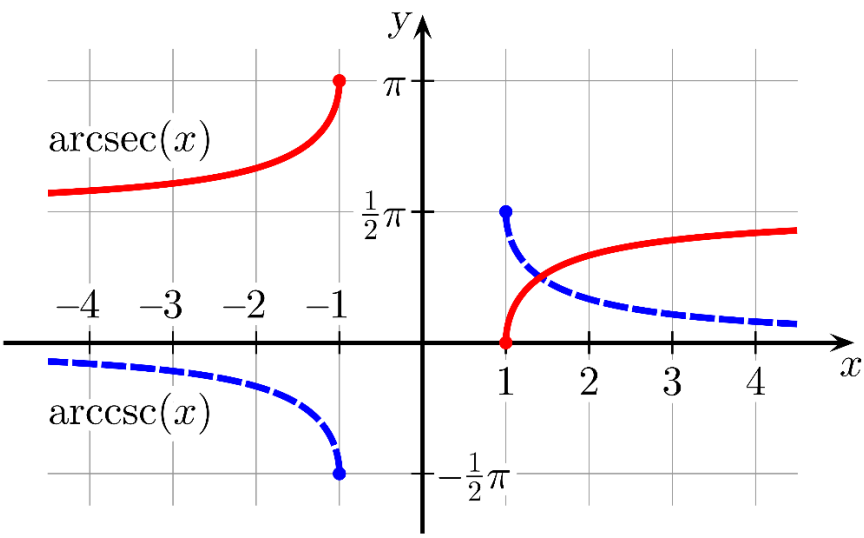


Image credits-

Graph of arcsin, arccos, arctan$\frac{y}{x}$: Desmos (desmos.com)

Unit circle of arcsin, arccos: Mathspace (mathspace.co)

Graph of arctan, arccot, arcsec, arccsc: ACADEMIC (en.academic.com)